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A Relation between the Anomalous Dimensions and OPE Coefficients in Asymptotic Free Field Theories*

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In asymptotic free field theories we show that part of the OPE of the trace of the stress-energy tensor and an arbitrary composite field is determined by the anomalous dimension of the composite field. We take examples from the two-dimensional O(N) non-linear sigma model.

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It is well known that in a renormalizable field theory the anomalous dimensions of composite fields determine the leading behavior of their OPE (operator product expansion) coefficients [1]. In this note we wish to show that part of the OPE of the stress-energy tensor and an arbitrary composite field is determined by the anomalous dimension of the composite field. Let us recall that in two dimensional conformal field theory the OPE of the stress-energy tensor and a conformal field is completely determined by the scale dimension of the conformal field [2]. The relation to be derived below is a weak generalization of this remarkable property of conformal field theory.

Let us consider an asymptotic free field theory with one dimensionless parameter g in the D-dimensional euclidean space. (Examples: g is the temperature for the O(N) non-linear sigma model in $D = 2$, and the strong fine structure constant for QCD without quarks in $D = 4$.) Let $\beta(g)$ be the beta function of the parameter:

$$\frac{d}{dt} g = \beta(g) \equiv \frac{1}{2}\ \beta_1 g^2 + \ldots\ . \tag{1}$$

Let $g(t)$ be the solution of the above RG (renormalization group) equation which satisfies the initial condition $g(t=0) = g$. With a spatial distance r we can form an RG invariant $g(\ln r)$. Note the asymptotic behavior

$$g(\ln r) \to \frac{1}{-\frac{\beta_1}{2}\ \ln r} \quad \text{as} \quad r \to 0\ . \tag{2}$$

Let Φ_a be a composite field with scale dimension $x_a + \gamma_a(g)$, where $\gamma_a(g) \equiv \gamma_{a,1} g + \ldots$ is the anomalous dimension. The field satisfies the RG equation

$$\frac{d}{dt}\Phi_a = (x_a + \gamma_a(g))\Phi_a\ , \tag{3}$$

where we assume no mixing for simplicity of the discussion.

We denote the trace of the stress-energy tensor by Θ which satisfies the canonical RG equation:

$$\frac{d}{dt}\Theta = D\Theta\ . \tag{4}$$

We consider the OPE of the trace Θ and an arbitrary composite field Φ_a:

$$\int_{|r|=\epsilon} d\Omega(r)\Theta(r)\Phi_a(0) \stackrel{\epsilon\to 0}{\longrightarrow} \mathcal{C}_a{}^b(\epsilon; g)\Phi_b(0)\ , \tag{5}$$

where $d\Omega$ is the $D-1$ dimensional angular volume element, and we have taken the angular average. We are only interested in the coefficients for which $x_b \le x_a$. The RG constrains the coefficients as follows:

$$\mathcal{C}_a{}^b(\epsilon; g) = \frac{1}{\epsilon^{1+x_a-x_b}} \exp\left[\int_{g(\ln\epsilon)}^{g} dx\ \frac{(\gamma_a - \gamma_b)(x)}{\beta(x)}\right] H_a{}^b(g(\ln\epsilon)) \ . \tag{6}$$

We wish to show that the leading behavior of the diagonal element $H_a{}^a(g)$ is given by

$$H_a{}^a(g) = \frac{\beta_1}{2}\ \gamma_{a,1}\ g^2 + \mathrm{O}(g^3) \ , \tag{7}$$

where $\gamma_{a,1}$ is the first Taylor coefficient of the anomalous dimension of Φ_a. This implies that

$$\mathcal{C}_a{}^a(\epsilon; g) = \frac{\gamma_{a,1}}{\frac{\beta_1}{2}\ \epsilon\ln^2\epsilon} + \mathrm{O}\left(\frac{1}{\epsilon\ln^3\epsilon}\right) \ . \tag{8}$$

To derive eqn. (7), we first recall that the volume integral of the trace Θ generates a scale transformation or equivalently an RG transformation [3]. Treating the short-distance singularities carefully, we obtain

$$\begin{aligned}
&\int_{|r-r_k|>\epsilon_k} d^D r\ \left\langle (\Theta(r) - \langle\Theta\rangle_g)\Phi_{a_1}(r_1)...\Phi_{a_n}(r_n)\right\rangle_g \\
\overset{\epsilon_k\to 0}{\longrightarrow}\quad & -\sum_{k=1}^{n}\left(x_{a_k} + r_{k\mu}\frac{\partial}{\partial r_{k\mu}}\right)\langle\Phi_{a_1}(r_1)...\Phi_{a_n}(r_n)\rangle_g \\
& -\sum_{k=1}^{n} S_{a_k}^{\ b}(\epsilon_k; g)\,\langle\Phi_{a_1}(r_1)...\Phi_b(r_k)...\Phi_{a_n}(r_n)\rangle_g \ .
\end{aligned} \tag{9}$$

By differentiating the above asymptotic expansion with respect to ϵ_k we obtain the relation of the coefficient $S_a{}^b$ to $\mathcal{C}_a{}^b$ in (5):

$$\frac{\partial}{\partial\epsilon} S_a{}^b(\epsilon; g) = \mathcal{C}_a{}^b(\epsilon; g) \ . \tag{10}$$

The RG constrains the coefficient $S_a{}^b$ in the form

$$S_a{}^b(\epsilon; g) = \frac{1}{\epsilon^{x_a-x_b}} \exp\left[\int_{g(\ln\epsilon)}^{g} dx\ \frac{(\gamma_a - \gamma_b)(x)}{\beta(x)}\right] \sigma_a{}^b(g(\ln\epsilon)) \ . \tag{11}$$

Hence, from eqs. (6), (10), and (11), we obtain

$$H_a{}^b(g) = \beta(g)\frac{d}{dg}\ \sigma_a{}^b(g) + (-x_a - \gamma_a(g) + x_b + \gamma_b(g))\sigma_a{}^b(g) \ . \tag{12}$$

We note that under the change of normalization

$$\Phi_a \to N_a(g)\Phi_a \ , \tag{13}$$

the matrix ${\sigma_a}^b$ in eqn. (11) changes homogeneously

$${\sigma_a}^b(g) \to N_a(g){\sigma_a}^b(g)(N_b(g))^{-1} \ . \tag{14}$$

Especially the diagonal element ${\sigma_a}^a$ is independent of the normalization. On the other hand, the anomalous dimension changes inhomogeneously

$$\gamma_a(g) \to \gamma_a(g) + \beta(g)\partial_g \ln N_a(g) \ . \tag{15}$$

If we allow $N_a(g)$ which either vanishes or diverges at $g = 0$, even the first Taylor coefficient $\gamma_{a,1}$ becomes normalization dependent.

We now choose to normalize Φ_a so that its two-point function has a non-vanishing limit as $g \to 0$:

$$\langle \Phi_a(r)\Phi_a(0)\rangle_{g\to 0} = \frac{\mathcal{N}_a}{r^{2x_a}} \ , \tag{16}$$

where $\mathcal{N}_a$ is a positive constant. Under this restriction, only such $N_a(g)$ which is finite and non-vanishing at $g = 0$ is allowed, and the first Taylor coefficient $\gamma_{a,1}$ becomes independent of normalization. With this convention, we now argue that

$$\gamma_a(g) = {\sigma_a}^a(g) + \mathrm{O}(g^2) \ . \tag{17}$$

To see this, we study the asymptotic expansion (9) perturbatively for an infinitesimal g for the case $n = 2$ and $a_1 = a_2 = a$. The trace Θ is proportional to the beta function β which is of order g^2, and we expect the right-hand side of (9) to vanish to order g^2. Because of the RG equation

$$\begin{aligned} &- \left(2x_a + r_\mu \frac{\partial}{\partial r_\mu}\right) \langle \Phi_a(r)\Phi_a(0)\rangle_g \\ &= \left(-\beta(g)\frac{\partial}{\partial g} + 2\gamma_a(g)\right) \langle \Phi_a(r)\Phi_a(0)\rangle_g \end{aligned} \tag{18}$$

and eq. (16), we find that the diagonal coefficient ${S_a}^a(\epsilon; g)$ must agree with the anomalous dimension $\gamma_a(g)$ to first order in g. Since

$${S_a}^a(\epsilon; g) = {\sigma_a}^a(g(\ln \epsilon)) \ , \tag{19}$$

we obtain eqn. (17). Finally, eqn. (7) follows from eqs. (12) and (17).

The matrix ${\sigma_a}^b$ have been calculated for the two-dimensional O(N) non-linear sigma model [4]. Here are three examples: ($\beta_1 = \frac{N-2}{\pi}$)
(i) The spin field $A \equiv \Phi^I (I = 1, ..., N)$ normalized by

$$\left\langle \Phi^I(r)\Phi^J(0) \right\rangle_{g\to 0} = \frac{1}{N}\delta^{IJ} \tag{20}$$

satisfies

$$\gamma_A(g) \simeq {\sigma_A}^A(g) \simeq \frac{N-1}{4\pi}\ g\ . \tag{21}$$

(ii) The first order derivative $B \equiv \frac{1}{\sqrt{g}}\partial_\mu \Phi^I$ satisfies

$$\left\langle \frac{1}{\sqrt{g}}\partial_\mu \Phi^I(r) \frac{1}{\sqrt{g}}\partial_\nu \Phi^J(0) \right\rangle_{g\to 0} = \delta^{IJ}\frac{1}{\pi r^2}\frac{N-1}{2N}(\delta_{\mu\nu} - 2\frac{r_\mu r_\nu}{r^2})\ , \tag{22}$$

and we find

$$\gamma_B(g) = \gamma_A(g) - \frac{\beta(g)}{2g} \simeq {\sigma_B}^B(g) \simeq \frac{1}{4\pi}\ g\ . \tag{23}$$

(iii) The second order derivative $C \equiv \frac{1}{g}\partial^2\Phi^I$ satisfies

$$\left\langle \frac{1}{g}\partial^2 \Phi^I(r) \frac{1}{g}\partial^2 \Phi^J(0) \right\rangle_{g\to 0} = \delta^{IJ}\frac{1}{\pi^2 r^4}\frac{N-1}{N}\ , \tag{24}$$

and we find

$$\gamma_C(g) = \gamma_A(g) - \frac{\beta(g)}{g} \simeq {\sigma_C}^C(g) \simeq \frac{-N+3}{4\pi}\ g\ . \tag{25}$$

In the above examples, the relation (17), which was missed in ref. [4], is verified explicitly.

Before concluding this letter, we consider yet another OPE:

$$\int_{|r|=\epsilon} d\Omega(r)\frac{r_\mu r_\nu}{\epsilon}\Theta_{\mu\nu}(r)\Phi_a(0) \stackrel{\epsilon\to 0}{\longrightarrow} {\tilde{\mathcal{C}}_a}{}^b(\epsilon; g)\Phi_b(0)\ , \tag{26}$$

where $\Theta_{\mu\nu}$ is the stress-energy tensor field which has no anomalous dimension. The RG implies

$${\tilde{\mathcal{C}}_a}{}^b(\epsilon; g) = \frac{1}{\epsilon^{x_a - x_b}} \exp\left[\int_{g(\ln\epsilon)}^{g} dx\ \frac{(\gamma_a - \gamma_b)(x)}{\beta(x)}\right] {\tilde{H}_a}{}^b(g(\ln\epsilon))\ . \tag{27}$$

Using the conservation law for the stress-energy tensor, we obtain

$$\partial_\mu(r_\nu \Theta_{\mu\nu}) = \Theta\ . \tag{28}$$

This implies

$$\frac{\partial}{\partial\epsilon}\int_{|r|=\epsilon} d\Omega(r)\frac{r_\mu r_\nu}{\epsilon}\Theta^{\mu\nu}(r)\Phi_a(0) = \int_{|r|=\epsilon} d\Omega(r)\Theta(r)\Phi_a(0) \ . \tag{29}$$

Hence,

$$\frac{\partial}{\partial\epsilon}\tilde{C}_a^{\ b}(\epsilon;g) = C_a^{\ b}(\epsilon;g) \ . \tag{30}$$

Comparing this with eqn. (10), we find that the difference between $\tilde{C}_a^{\ b}(\epsilon;g)$ and $S_a^{\ b}(\epsilon;g)$ is independent of ϵ. Therefore, using the RG constraints (11) and (27), we obtain

$$\tilde{H}_a^{\ b}(g) = \sigma_a^{\ b}(g) + k_a\delta_a^{\ b} \ , \tag{31}$$

where k_a is a constant. Since the constant k_a is the value of $\tilde{H}_a^{\ a}(g)$ at $g=0$, we can resort to the free theory at $g=0$ for its determination. Then we find

$$k_a = x_a \ , \tag{32}$$

where x_a is the naïve scale dimension of Φ_a. Thus, the leading behavior of the diagonal term $\tilde{C}_a^{\ a}$ is given by

$$\tilde{C}_a^{\ a}(\epsilon;g) = x_a + \frac{\gamma_{a,1}}{-\frac{\beta_1}{2}\ \ln\epsilon} + \mathrm{O}\left(\frac{1}{\ln^2\epsilon}\right) \ . \tag{33}$$

In conclusion we have derived the following leading behavior of the OPE's in asymptotic free field theories:

$$\begin{aligned} &\int_{|r|=\epsilon} d\Omega(r)\Theta(r)\Phi_a(0) \to \frac{\gamma_{a,1}}{\frac{\beta_1}{2}\ \epsilon\ln^2\epsilon}\Phi_a(0) + \text{off diagonal terms} \\ &\int_{|r|=\epsilon} d\Omega(r)\frac{r_\mu r_\nu}{\epsilon}\ \Theta^{\mu\nu}(r)\Phi_a(0) \to \left(x_a + \frac{\gamma_{a,1}}{-\frac{\beta_1}{2}\ln\epsilon}\right)\Phi_a(0) + \text{off diagonal terms} \ , \end{aligned} \tag{34}$$

where $\gamma_{a,1}$ is the first Taylor coefficient of the anomalous dimension of the field Φ_a which is normalized so that its two-point function has a non-vanishing limit at $g=0$ (eqn. (16)). The above OPE's constitute a weak generalization of the structure of the OPE of the stress-energy tensor and an arbitrary conformal field in two dimensional conformal field theory. The extension of eqs. (34) to asymptotic free field theories with more than one parameter is straightforward.

Analogous results for the equal-time commutator of Θ and the elementary field ϕ in the perturbative ϕ^4 theory was obtained earlier in refs. [3] and [5], and more recently in ref. [6]. Here, we emphasize the importance of normalizing the composite fields properly (see (16)) to get the above results (34).

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